

# Formulations and relaxations for a multi-echelon capacitated location–distribution problem

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## Abstract

We consider a multi-echelon location–distribution problem arising from an actual application in fast delivery service. We present and compare two formulations for this problem: an arc-based model and a path-based model. We show that the linear programming (LP) relaxation of the path-based model provides a better bound than the LP relaxation of the arc-based model. We also compare the so-called binary relaxations of the models, which are obtained by relaxing the integrality constraints for the general integer variables, but not for the 0–1 variables. We show that the binary relaxations of the two models always provide the same bound, but that the path-based binary relaxation appears preferable from a computational point of view, since it can be reformulated as an equivalent simple plant location problem (SPLP), for which several efficient algorithms exist. We also show that the LP relaxation of this SPLP reformulation provides a better bound than the LP relaxation of the path-based model.

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## 1. Introduction

In this paper, we consider a multi-echelon location–distribution problem arising from an actual application in fast delivery service, first described by Gendron et al. [1]: a mail-order company offers several products (typically, packages containing various types of goods, such as clothes, electronic devices, appliances, . . .) that must be delivered on time to the customers requesting them. To satisfy these requests, the firm operates a multi-echelon distribution system: starting their trips from a small set of hubs (their locations are assumed known and fixed, following a preliminary strategic analysis), a fleet of medium-size trucks delivers the products to depots, where they are transferred on small-size trucks, and then shipped to satellites, where the products are sorted and delivered to the customers. The company exploits existing facilities for the depots and the satellites, but has to pay to use them. The problem is to ensure that customers' requests are satisfied on time at minimum cost, taking into account the transportation costs and the location costs for using the depots and the satellites.

Fig. 1 illustrates a typical network representing this multi-echelon location–distribution system. Starting their trips from a single hub  $h$ , the products must be delivered to three customers,  $l$ ,  $l'$  and  $l''$ . From hub  $h$ , medium-size trucks

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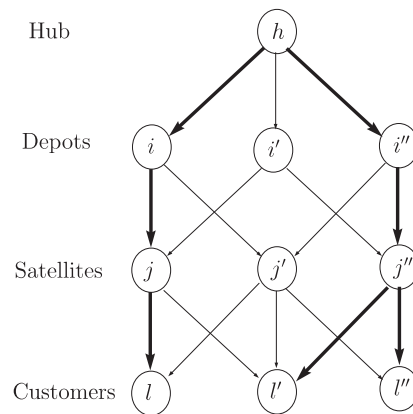


Fig. 1. Typical multi-echelon location-distribution network.

are used to ship the products to depots (up to three depots,  $i$ ,  $i'$  and  $i''$ , are available), where the products are transferred on small-size trucks and delivered to satellites (there are three satellites,  $j$ ,  $j'$  and  $j''$ ). The arcs in bold represent the links used to transport the products in a solution to the problem, where only two depots ( $i$  and  $i''$ ) and two satellites ( $j$  and  $j''$ ) are used.

Typically, satellites are neither owned nor rented by the company. They can be warehouses owned by independent carriers or sites such as car parks where items are transferred from one vehicle to another. More traditional distribution systems, which do not use such satellites, can be very costly to run when the demand varies significantly from one period to another. Moreover, when few depots are present, services such as 24 h-delivery can be cost ineffective or simply cannot be assured when a wide distribution area is considered. The multi-echelon system we consider allows to satisfy time constraints in a cost-effective way. It is also an *adaptive* system, in the sense that satellites can be opened or closed easily according to demand variations.

We model the problem by defining a network for which the only possible connections are those that ensure on time delivery of the products to the customers. In addition, we assume that for each satellite and each product, the set of customers and the routes used to satisfy their requests have been determined in a preprocessing phase. Hence, the model does not include any routing aspect. Each customer in our model represents a set of customers to which the same product is delivered using a single vehicle. The transportation cost between a satellite and a customer thus corresponds to the cost of the best route determined during this preprocessing phase.

Transportation costs between hubs and depots, and between depots and satellites, vary with the distance travelled, but more importantly, with the number of vehicles used on each arc, each type of vehicle (medium- or small-size truck) having an associated volumetric capacity. A fixed cost is incurred when using any depot, while the satellite location cost increases with the number of batches of products handled at the satellite (a batch corresponds to a fixed number of product units). This cost structure is similar to what can be found in telecommunications network design applications, where multiple facilities, each with an associated capacity, can be installed on the arcs or the nodes. In our problem, vehicles (at the arcs) and product batches (at the satellites) play the role of facilities. Note that this cost structure is more complex than what can be found in most location-distribution problems discussed in the literature, which typically exhibit fixed costs at the nodes and transportation costs that are linear in the number of product units. Nevertheless, location-distribution problems with this type of cost functions have been studied in the past, as evidenced by recent contributions by Correia and Captivo [2], Melo et al. [3], and the references therein.

Because the locations of the hubs are assumed to be fixed, there are no fixed costs associated to the hubs and we can always assign to each depot its closest hub without losing optimality. This simplification is also performed in the preprocessing phase. Note that we still need to determine how many product units, on how many medium-size vehicles, need to be transported between any depot and its closest hub, but we are now allowed to associate the corresponding decision variables to the depots, instead of the arcs between hubs and depots. The resulting problem can therefore be considered as a *two-echelon* (from depots to satellites, and from satellites to customers) capacitated location-distribution problem.

We present and compare two mixed-integer programming (MIP) formulations for this problem: an arc-based model and a path-based model. We show that the linear programming (LP) relaxation of the path-based model provides a better bound than the LP relaxation of the arc-based model. We also compare the so-called *binary relaxations* of the models, which are obtained by relaxing the integrality constraints for all variables, except for the 0–1 design variables that determine which nodes and which arcs should be used to satisfy customers' requests. We show that the binary relaxations of the two models always provide the same bound, but that the path-based binary relaxation appears preferable from a computational point of view, since it can be reformulated as an equivalent *simple plant location problem* (SPLP), for which several efficient algorithms exist (such as the classical dual-ascent method of Erlenkotter [4] and the recent variable neighborhood search approach of Hansen [5]). We also show that the LP relaxation of this SPLP reformulation provides a better bound than the LP relaxation of the path-based model. Finally, we present computational results on solving these different formulations and relaxations with the state-of-the-art LP/MIP solver CPLEX (version 10.0) on an actual application and on instances derived from this large-scale real network.

There is an abundant literature on two-echelon location–distribution problems. In particular, several authors study relaxations for either an arc-based formulation [6–8] or a path-based formulation [9–12], or compare relaxations for the two formulations [13–15]. The models we study differ significantly from the ones considered in the literature, because the objective function in our formulations involves facility installation costs at the arcs (represented by the number of vehicles used on the arcs) and at some of the nodes (corresponding to the number of product batches at the satellites). In addition, there are no fixed costs at the intermediate locations (here, the satellites), contrary to what can be found in the existing literature. This is an important difference, as the absence of fixed costs at the satellites explains why the binary relaxation of the path-based model can be reformulated as an SPLP. In Section 6, we further comment on this issue.

The paper is organized as follows. In Section 2, we introduce the problem notation and a few basic definitions that will be useful when comparing the models. In the next two sections, 3 and 4, we present the arc-based and the path-based models, respectively. In Section 5, we compare the relaxations of the two models. In Section 6, we show that the path-based binary relaxation can be reformulated as an SPLP, and that the LP relaxation of this SPLP reformulation is better than the LP relaxation of the path-based model. All bound relationships between the different models and their relaxations are summarized in Section 7. Section 8 presents computational results on solving instances derived from the large-scale application that motivated this study. We conclude with the presentation of some avenues for future research.

## 2. Notation and definitions

The following sets define the different types of nodes in the network:

$D$  set of potential sites to locate depots;

$S$  set of potential sites to locate satellites;

$L$  set of customers;

$D_j^S$  set of potential sites to locate depots connected to satellite  $j \in S$ ;

$S_i^D$  set of potential sites to locate satellites connected to depot site  $i \in D$ ;

$S_l^L$  set of potential sites to locate satellites connected to customer  $l \in L$ ;

$L_j^S$  set of customers connected to satellite  $j \in S$ ;

$L_i^D$  set of customers connected to depot  $i \in D$  from some satellite  $j \in S_i^D$ .

The data related to the customer demands and the vehicle capacities are defined as follows (all values are assumed to be positive):

$n_l$  number of product units to deliver to customer  $l \in L$ ;

$v_l$  volume of product units to deliver to customer  $l \in L$ ;

$Q$  capacity (in number of product units) of one batch of products handled at any satellite;

$P$  volumetric capacity of a medium-size vehicle transporting product units to any depot, from its closest hub;

$R$  volumetric capacity of a small-size vehicle transporting product units from any depot to any satellite.

The location and transportation costs are defined as follows (all values are assumed to be nonnegative):

$f_i$  fixed cost for using and operating depot  $i \in D$ ;

$g_j$  cost per  $Q$  product units for using and operating satellite  $j \in S$ ;

$d_i$  transportation cost for using one medium-size vehicle to transport product units to depot  $i \in D$  from its closest hub;

$e_{ij}$  transportation cost for using one small-size vehicle from depot  $i \in D$  to satellite  $j \in S_i^D$ ;

$c_{jl}$  transportation cost between satellite  $j \in S$  and customer  $l \in L_j^S$ .

The models we propose are MIP formulations having three types of variables: continuous, binary and general integer variables. When relaxing all integrality requirements (on binary and general integer variables), we obtain the LP relaxation. For any model  $M$ , we will denote its LP relaxation as  $LP(M)$ . If we relax the integrality constraints on the general integer variables only, we obtain the binary relaxation, which is a mixed 0–1 programming formulation. For any model  $M$ , we will denote its binary relaxation as  $BIN(M)$ .

We will use the following definitions to compare different relaxations for the same minimization problem, where  $Z(M)$  denote the optimal objective value of any model  $M$ :

**Definition 1.** Relaxation  $M_1$  is *equivalent* to relaxation  $M_2$  if  $Z(M_1) = Z(M_2)$  for all problem instances.

Note that a model is a relaxation of itself, so we can use the same definition to show that two models are equivalent.

**Definition 2.** Relaxation  $M_1$  *dominates* relaxation  $M_2$  if  $Z(M_1) \geq Z(M_2)$  for all problem instances and there exists at least one problem instance such that  $Z(M_1) > Z(M_2)$ .

### 3. Arc-based formulation

To derive the arc-based model, the following sets of binary variables are introduced:

$$X_{jl} = \begin{cases} 1 & \text{if some product units are transported between satellite } j \in S \\ & \text{and customer } l \in L_j^S; \\ 0 & \text{otherwise;} \end{cases}$$

$$W_{ij} = \begin{cases} 1 & \text{if some product units are transported between depot } i \in D \\ & \text{and satellite } j \in S_i^D; \\ 0 & \text{otherwise;} \end{cases}$$

$$Y_i = \begin{cases} 1 & \text{if some product units are transported to depot } i \in D \\ & \text{from its closest hub;} \\ 0 & \text{otherwise.} \end{cases}$$

In addition, we introduce flow variables representing the volume of product units transported between depots and satellites:

$$V_{ij} = \text{volume of product units transported between depot } i \in D \text{ and satellite } j \in S_i^D.$$

We also use the following general integer variables to represent the number of batches handled at any satellite and the number of vehicles used on any depot-satellite arc or at any depot:

$U_j$  = number of batches of products handled at satellite  $j \in S$ ;

$T_i$  = number of medium-size vehicles used between depot  $i \in D$  and its closest hub;

$H_{ij}$  = number of small-size vehicles used between depot  $i \in D$  and satellite  $j \in S_i^D$ .

The arc-based formulation of the problem, denoted  $M_{\text{arc}}$ , can then be written as follows:

$$\min \sum_{i \in D} f_i Y_i + \sum_{j \in S} g_j U_j + \sum_{i \in D} d_i T_i + \sum_{i \in D} \sum_{j \in S_i^D} e_{ij} H_{ij} + \sum_{j \in S} \sum_{l \in L_j^S} c_{jl} X_{jl}, \quad (1)$$

$$\sum_{j \in S_l^L} X_{jl} = 1, \quad \forall l \in L, \quad (2)$$

$$\sum_{i \in D_j^S} W_{ij} \leq 1, \quad \forall j \in S, \tag{3}$$

$$\sum_{i \in D_j^S} V_{ij} = \sum_{l \in L_j^S} v_l X_{jl}, \quad \forall j \in S, \tag{4}$$

$$V_{ij} \leq \left( \sum_{l \in L_j^S} v_l \right) W_{ij}, \quad \forall i \in D, \quad \forall j \in S_i^D, \tag{5}$$

$$\sum_{j \in S_i^D} V_{ij} \leq \left( \sum_{l \in L_i^D} v_l \right) Y_i, \quad \forall i \in D, \tag{6}$$

$$X_{jl} \leq \sum_{i \in D_j^S} W_{ij}, \quad \forall j \in S, \quad \forall l \in L_j^S, \tag{7}$$

$$W_{ij} \leq Y_i, \quad \forall i \in D, \quad \forall j \in S_i^D, \tag{8}$$

$$\sum_{l \in L_j^S} n_l X_{jl} \leq QU_j, \quad \forall j \in S, \tag{9}$$

$$\sum_{j \in S_i^D} V_{ij} \leq PT_i, \quad \forall i \in D, \tag{10}$$

$$V_{ij} \leq RH_{ij}, \quad \forall i \in D, \quad \forall j \in S_i^D, \tag{11}$$

$$U_j \geq 0 \text{ and integer}, \quad \forall j \in S, \tag{12}$$

$$T_i \geq 0 \text{ and integer}, \quad \forall i \in D, \tag{13}$$

$$H_{ij} \geq 0 \text{ and integer}, \quad \forall i \in D, \quad \forall j \in S_i^D, \tag{14}$$

$$X_{jl} \in \{0, 1\}, \quad \forall j \in S, \quad \forall l \in L_j^S, \tag{15}$$

$$W_{ij} \in \{0, 1\}, \quad \forall i \in D, \quad \forall j \in S_i^D, \tag{16}$$

$$Y_i \in \{0, 1\}, \quad \forall i \in D, \tag{17}$$

$$V_{ij} \geq 0, \quad \forall i \in D, \quad \forall j \in S_i^D. \tag{18}$$

The objective function, (1), consists in minimizing all costs incurred by using and operating depots and satellites, as well as transportation costs between hubs and depots, between depots and satellites, and between satellites and customers. Constraints (2) ensure that each customer is being served by a single satellite. Constraints (3) ensure that any satellite, when it is used, is connected to a single depot. Constraints (4) are flow conservation equations at each satellite. The forcing constraints (5) and (6) link together the flow variables and the binary variables: they ensure that no flow can circulate on a network element (depot-satellite arc or depot) that is not used to transport product units. Constraints (7) and (8) are also forcing constraints that link together the different types of binary variables. Constraints (7) ensure that any customer cannot be routed from a satellite that is not connected to some depot. Similarly, constraints (8) ensure that any satellite cannot be connected to a depot that is not used to transport product units. Constraints (9) ensure that the number of product units handled at a satellite cannot exceed the capacity of product batches. Constraints (10) and (11) ensure that the total volume of all product units transported on a network element (depot or depot-satellite arc) cannot exceed the capacity of the vehicles used on that network element. Other constraints specify the nature of the different types of variables.

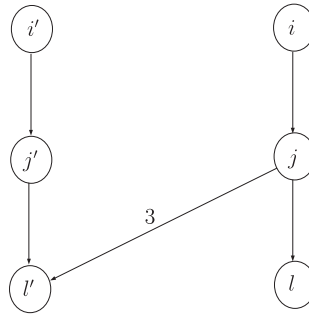


Fig. 2. Example showing that constraints (7) and (8) are not redundant in  $LP(M_{\text{arc}})$ .

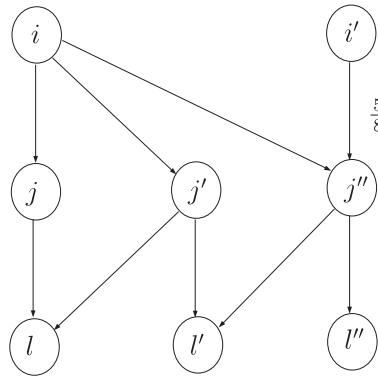


Fig. 3. Example showing that  $LP(M_{\text{arc}})$  can have an optimal fractional solution.

It is easy to show that the forcing constraints (7) and (8) are redundant. However, it can be beneficial to keep them in the model, since they might improve the lower bounds obtained from the LP relaxation, as shown by the example illustrated in Fig. 2, which has two depots,  $i$  and  $i'$ , two satellites,  $j$  and  $j'$ , and two customers,  $l$  and  $l'$ . We assume that all parameter values (volume and number of product units requested by the customer, capacities and costs) are equal to 1, except  $c_{j'l'}$  which is equal to 3. There are two feasible solutions to model  $M_{\text{arc}}$ :

- (1) Send one product unit on path  $(i, j, l)$  and one product unit on path  $(i, j, l')$ , with an objective value equal to  $f_i + 2 \times g_j + 2 \times d_i + 2 \times e_{ij} + c_{jl} + c_{j'l'} = 11$ .
- (2) Send one product unit on path  $(i, j, l)$  and one product unit on path  $(i', j', l')$ , with an objective function value equal to  $f_i + f_{i'} + g_j + g_{j'} + d_i + d_{i'} + e_{ij} + e_{i'j'} + c_{jl} + c_{j'l'} = 10$ , so this is the optimal solution.

An optimal solution to  $LP(M_{\text{arc}})$  follows the same paths to transport product units as in this last solution. In addition, we have  $X_{jl} = 1$  and, by (7),  $W_{ij} = 1$ . Finally, because of constraints (8), we conclude that  $Y_i = 1$ . Hence, the optimal solution to  $LP(M_{\text{arc}})$  is the same as that of  $M_{\text{arc}}$  and  $Z(LP(M_{\text{arc}})) = 10$ . If we remove constraints (8) from  $LP(M_{\text{arc}})$ , we obtain a different conclusion: because  $V_{ij} = 1$  and  $\sum_{l \in L_i^D} v_l = 2$ , we have  $Y_i = \frac{1}{2}$ , by (6). Similarly, if we keep constraints (8), but we remove (7) from  $LP(M_{\text{arc}})$ , we obtain  $W_{ij} = \frac{1}{2}$  by (5), since  $V_{ij} = 1$  and  $\sum_{l \in L_j^D} v_l = 2$ . In this case, we also have  $Y_i = \frac{1}{2}$ . Therefore, if we remove (7) or (8) from the LP relaxation, the optimal objective value is equal to  $9\frac{1}{2} < Z(LP(M_{\text{arc}}))$ .

In the last example, the solution to  $LP(M_{\text{arc}})$  is the same as that of  $M_{\text{arc}}$ . This is not always the case, as illustrated by the problem instance of Fig. 3, which has two depots,  $i$  and  $i'$ , three satellites,  $j, j'$  and  $j''$ , and three customers,  $l, l'$  and  $l''$ . We assume that all parameter values (volume and number of product units requested by the customer, capacities and costs) are equal to 1, except  $e_{i'j''}$  which is equal to  $\frac{5}{8}$ . Note that, if we disregard the fixed charges at the depots, the

cost of sending one product on any path originating at  $i$  and  $i'$  is equal to 4 and  $3\frac{5}{8}$ , respectively. An optimal solution to this problem consists in sending one product unit on each of the three paths  $(i, j, l)$ ,  $(i, j', l')$  and  $(i, j'', l'')$ , with an objective value equal to 13. An optimal solution to  $LP(M_{\text{arc}})$  splits the flow evenly on the four paths connecting  $i$  to  $l$  and  $l'$ , and sends one product unit on the path from  $i'$  to  $l''$ :  $V_{ij} = V_{ij''} = \frac{1}{2}$  and  $V_{i'j'} = V_{i'l''} = 1$ . As a result, all  $W$  variables assume value  $\frac{1}{2}$ , and we conclude that  $Y_i = Y_{i'} = \frac{1}{2}$ . Hence, the optimal objective value of  $LP(M_{\text{arc}})$  is equal to  $\frac{1}{2} \times f_i + \frac{1}{2} \times f_{i'} + 4 \times (\frac{1}{2} \times 4) + 3\frac{5}{8} = 12\frac{5}{8} < Z(M_{\text{arc}})$ .

The reason for obtaining such fractional solutions in  $LP(M_{\text{arc}})$  lies in the weakness of the forcing constraints (7), which are defined by using sums of binary variables, instead of individual binary variables. The path-based formulation, which we present next, is an equivalent MIP model that provides a better LP relaxation by introducing tighter forcing constraints.

#### 4. Path-based formulation

To define the path-based model, the following set of binary variables is introduced:

$$X_{ijl} = \begin{cases} 1 & \text{if some product units are transported on path } (i, j, l), \\ & i \in D, j \in S_i^D, l \in L_j^S; \\ 0 & \text{otherwise.} \end{cases}$$

These variables are linked to the variables of the arc-based model by the following equations:

$$X_{jl} = \sum_{i \in D_j^S} X_{ijl}, \quad \forall j \in S, \forall l \in L_j^S, \tag{19}$$

$$V_{ij} = \sum_{l \in L_j^S} v_l X_{ijl}, \quad \forall i \in D, \forall j \in S_i^D. \tag{20}$$

We can project out of model  $M_{\text{arc}}$  the variables defined by these equations and remove the flow conservation equations (4), which become redundant. In addition, we replace the weak forcing constraints (7) by stronger ones which make use of tighter links between the binary variables. We then obtain the path-based formulation, denoted  $M_{\text{path}}$ :

$$\min \sum_{i \in D} f_i Y_i + \sum_{j \in S} g_j U_j + \sum_{i \in D} d_i T_i + \sum_{i \in D} \sum_{j \in S_i^D} e_{ij} H_{ij} + \sum_{i \in D} \sum_{j \in S_i^D} \sum_{l \in L_j^S} c_{jl} X_{ijl}, \tag{21}$$

$$\sum_{j \in S_i^L} \sum_{i \in D_j^S} X_{ijl} = 1, \quad \forall l \in L, \tag{22}$$

$$\sum_{i \in D_j^S} W_{ij} \leq 1, \quad \forall j \in S, \tag{23}$$

$$\sum_{j \in S_i^D} \sum_{l \in L_j^S} v_l X_{ijl} \leq \left( \sum_{l \in L_i^D} v_l \right) Y_i, \quad \forall i \in D, \tag{24}$$

$$X_{ijl} \leq W_{ij}, \quad \forall i \in D, \forall j \in S_i^D, \forall l \in L_j^S, \tag{25}$$

$$W_{ij} \leq Y_i, \quad \forall i \in D, \forall j \in S_i^D, \tag{26}$$

$$\sum_{i \in D_j^S} \sum_{l \in L_j^S} n_l X_{ijl} \leq QU_j, \quad \forall j \in S, \tag{27}$$

$$\sum_{j \in S_i^D} \sum_{l \in L_j^S} v_l X_{ijl} \leq PT_i, \quad \forall i \in D, \quad (28)$$

$$\sum_{l \in L_j^S} v_l X_{ijl} \leq RH_{ij}, \quad \forall i \in D, \quad \forall j \in S_i^D, \quad (29)$$

$$U_j \geq 0 \text{ and integer}, \quad \forall j \in S, \quad (30)$$

$$T_i \geq 0 \text{ and integer}, \quad \forall i \in D, \quad (31)$$

$$H_{ij} \geq 0 \text{ and integer}, \quad \forall i \in D, \quad \forall j \in S_i^D, \quad (32)$$

$$X_{ijl} \in \{0, 1\}, \quad \forall i \in D, \quad \forall j \in S_i^D, \quad \forall l \in L_j^S, \quad (33)$$

$$W_{ij} \in \{0, 1\}, \quad \forall i \in D, \quad \forall j \in S_i^D, \quad (34)$$

$$Y_i \in \{0, 1\}, \quad \forall i \in D. \quad (35)$$

**Proposition 3.**  $M_{\text{arc}}$  and  $M_{\text{path}}$  are equivalent.

**Proof.** (1)  $Z(M_{\text{arc}}) \geq Z(M_{\text{path}})$

Consider an optimal solution to  $M_{\text{arc}}$  and define

$$X_{ijl} = \begin{cases} 1 & \text{if } X_{jl} = 1 \text{ and } W_{ij} = 1, i \in D, j \in S_i^D, l \in L_j^S; \\ 0 & \text{otherwise.} \end{cases} \quad (36)$$

An equivalent definition is  $X_{ijl} = W_{ij}X_{jl}$ ,  $i \in D$ ,  $j \in S_i^D$ ,  $l \in L_j^S$ , which implies the two following equations:

- (a)  $\sum_{i \in D_j^S} X_{ijl} = (\sum_{i \in D_j^S} W_{ij})X_{jl} = X_{jl}$ ,  $j \in S$ ,  $l \in L_j^S$  (the last equation follows immediately from constraints (3) and (7));
- (b)  $\sum_{l \in L_j^S} v_l X_{ijl} = W_{ij} \sum_{l \in L_j^S} v_l X_{jl} = W_{ij} \sum_{i' \in D_j^S} V_{i'j} = V_{ij}$ ,  $i \in D$ ,  $j \in S_i^D$  (the last equation follows immediately from constraints (3) and (5)).

Now, we can derive a solution to  $M_{\text{path}}$  by using definition (36), all other variables in  $M_{\text{path}}$  having the same values as in the optimal solution to  $M_{\text{arc}}$ ; this solution is feasible since:

constraints (23) and (26) are satisfied, since they are the same as (3) and (8), respectively;

(36) implies constraints (25);

by using equations (a), we derive constraints (22) and (27) from (2) and (9), respectively;

by using equations (b), we derive constraints (24), (29) and (28) from (6), (11) and (10), respectively.

By equations (a), this solution has the same objective value as the optimal solution to  $M_{\text{arc}}$ , which shows that  $Z(M_{\text{arc}}) \geq Z(M_{\text{path}})$ .

(2)  $Z(M_{\text{path}}) \geq Z(M_{\text{arc}})$ .

Consider an optimal solution to  $M_{\text{path}}$  and derive a solution to  $M_{\text{arc}}$  by using equations (19) and (20), all other variables assuming the same values as in the optimal solution to  $M_{\text{path}}$ ; this solution is feasible since:

constraints (3) and (8) are satisfied, since they are the same as (23) and (26), respectively;

by using Eqs. (19), we derive constraints (2) and (9) from (22) and (27);

by using Eqs. (20), we derive constraints (6), (11) and (10) from (24), (29) and (28);

by combining Eqs. (19) and (20), we obtain (4);

by summing (25) over  $i \in D$ , we obtain constraints (7) from Eq. (19);

by multiplying (25) by  $v_l$  and by summing them over  $l \in L_j^S$ , we obtain constraints (5).

By Eqs. (19), this solution has the same objective value as the optimal solution to  $M_{\text{path}}$ , which shows that  $Z(M_{\text{path}}) \geq Z(M_{\text{arc}})$ .  $\square$



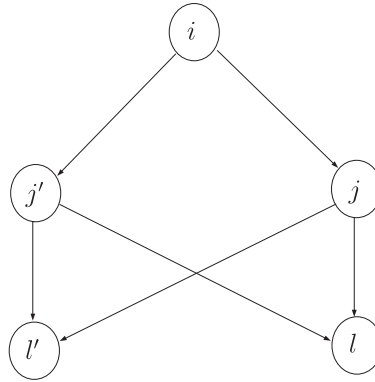


Fig. 4. Example showing that constraints (24) are not redundant in  $LP(M_{\text{path}})$ .

Note that model  $M_{\text{path}}$  contains no constraints corresponding to the forcing constraints (5) of  $M_{\text{arc}}$ . As we have just seen in the proof of Proposition 3, the latter are obtained by aggregation of the forcing constraints (25). Constraints (24), however, cannot be obtained by simple aggregation of the other forcing constraints and, although they are redundant in model  $M_{\text{path}}$ , they are not in its LP relaxation, as shown by the problem instance displayed in Fig. 4, which has one depot,  $i$ , two satellites,  $j$  and  $j'$ , and two customers,  $l$  and  $l'$ . We assume that all parameter values (volume and number of product units requested by the customer, capacities and costs) are equal to 1. An optimal solution to  $LP(M_{\text{path}})$  splits the flow evenly on all possible paths:  $X_{ijl} = X_{ijj'l} = X_{ij'l} = X_{ij'l'} = \frac{1}{2}$ . Because of the forcing constraints (25), we have  $W_{ij} = W_{ijj'} = \frac{1}{2}$ . Now, if only constraints (26) were present, we would have  $Y_i = \frac{1}{2}$ , but because of constraints (24), we have  $Y_i = 1$ , since  $\sum_{l \in L_i^D} v_l = 2$  and  $\sum_{j \in S_i^D} \sum_{l \in L_j^S} v_l X_{ijl} = 2$ .

The situation is the same the other way around: if we keep constraints (24) in the formulation, then constraints (26) are redundant in the model, but not in its LP relaxation. To verify this result, consider again the problem instance shown in Fig. 2. An optimal solution to  $LP(M_{\text{path}})$  follows the paths  $(i, j, l)$  and  $(i', j', l')$ . We also have  $X_{ijl} = X_{i'j'l'} = W_{ij} = W_{i'j'l'} = Y_{i'} = 1$ , but  $Y_i = \frac{1}{2}$ , since  $\sum_{l \in L_i^D} v_l = 2$  and only one product unit is transported on path  $(i, j, l)$ . By adding constraints (26) to the LP relaxation, we would have obtained  $Y_i = 1$ .

For the problem instances of Figs. 2 and 4, the LP relaxation of  $M_{\text{path}}$  solves with 0–1 values for all binary variables. Unfortunately, this is not always the case. To show this, consider again the problem instance illustrated in Fig. 3. As we already know,  $Z(M_{\text{path}}) = Z(M_{\text{arc}}) = 13$  for this problem instance. An optimal solution to  $LP(M_{\text{path}})$  splits the flow evenly on the two paths from  $i$  to  $l$ , and sends one product unit on each of the two paths originating at  $i'$ :  $X_{ijl} = X_{ij'l} = \frac{1}{2}$  and  $X_{i'j''l'} = X_{i'j''l''} = 1$ . As a result, we have  $W_{ij} = W_{ijj'} = Y_i = \frac{1}{2}$  and  $W_{i'j''} = Y_{i'} = 1$ . Hence, the optimal objective value of  $LP(M_{\text{path}})$  is equal to  $\frac{1}{2} \times f_i + f_{i'} + 2 \times (\frac{1}{2} \times 4) + 2 \times (3 \frac{5}{8}) = 12 \frac{3}{4} < Z(M_{\text{path}})$ .

This example also shows that we can have  $Z(LP(M_{\text{path}})) > Z(LP(M_{\text{arc}}))$ . This happens because, from an optimal solution to  $LP(M_{\text{arc}})$ , we cannot derive a feasible solution to  $LP(M_{\text{path}})$  as in the proof of Proposition 3. Indeed, in an optimal solution to  $LP(M_{\text{arc}})$  for the problem instance illustrated in Fig. 3, path  $(i, j'', l')$  is used to transport  $\frac{1}{2}$  product unit and path  $(i', j'', l'')$  is used to transport 1 product unit. If we try to derive a feasible solution to  $LP(M_{\text{path}})$  from this solution, this would imply  $W_{ij''} \geq X_{ij''l'} = \frac{1}{2}$  and  $W_{i'j''} \geq X_{i'j''l''} = 1$ , hence  $W_{ij''} + W_{i'j''} = \frac{3}{2} > 1$  and constraints (23) would be violated. The next section further clarifies the relationships between the relaxations of the two formulations.

### 5. Comparison of relaxations

The introduction of the path-based variables is motivated by the goal of deriving a better LP relaxation than the one obtained from the arc-based formulation. This property is stated and proved in the following proposition.

**Proposition 4.**  $LP(M_{\text{path}})$  dominates  $LP(M_{\text{arc}})$ .

**Proof.** To show that  $Z(LP(M_{\text{path}})) \geq Z(LP(M_{\text{arc}}))$ , we use exactly the same arguments as in the second part of the proof of Proposition 3, where it was shown that  $Z(M_{\text{path}}) \geq Z(M_{\text{arc}})$ .

As we have seen at the end of the previous section, Fig. 3 provides a problem instance such that  $Z(LP(M_{\text{path}})) > Z(LP(M_{\text{arc}}))$ .  $\square$

An interesting alternative to the LP relaxation is the binary relaxation, which is obtained by relaxing the integrality constraints for the general integer variables only, but not for the binary variables. The resulting formulation is a 0–1 MIP problem, having only continuous and binary variables. We then have the following proposition, the proof of which is omitted, since it follows exactly the same arguments as the proof of Proposition 3:

**Proposition 5.**  *$BIN(M_{\text{arc}})$  and  $BIN(M_{\text{path}})$  are equivalent.*

If we are to use the binary relaxation as a basis for solution algorithms, this proposition implies that we cannot favor formulation  $M_{\text{path}}$  over  $M_{\text{arc}}$  based on the quality of the lower bound. So, at first sight, it appears that model  $BIN(M_{\text{arc}})$  is more promising as it involves much less variables. If we look more closely into the structure of  $BIN(M_{\text{arc}})$ , we first have the following proposition that allows to further reduce the number of variables.

**Proposition 6.** *There exists an optimal solution to  $BIN(M_{\text{arc}})$  such that*

$$U_j = \frac{1}{Q} \sum_{l \in L_j^S} n_l X_{jl}, \quad \forall j \in S, \tag{37}$$

$$T_i = \frac{1}{P} \sum_{j \in S_i^D} V_{ij}, \quad \forall i \in D, \tag{38}$$

$$H_{ij} = \frac{1}{R} V_{ij}, \quad \forall i \in D, \forall j \in S_i^D. \tag{39}$$

**Proof.** Constraints (9)–(11) define lower bounds on the values of each of the variables  $U_j$ ,  $T_i$  and  $H_{ij}$ , respectively. Since these variables do not appear in any other constraints and the costs  $g_j$ ,  $d_i$  and  $e_{ij}$  are all nonnegative, these variables (now continuous in model  $BIN(M_{\text{arc}})$ ) must assume these lower bounds in some optimal solution.  $\square$

By using this proposition, we can reformulate  $BIN(M_{\text{arc}})$  as follows:

$$\min \sum_{i \in D} f_i Y_i + \sum_{j \in S} \sum_{l \in L_j^S} \left( c_{jl} + n_l \frac{g_j}{Q} \right) X_{jl} + \sum_{i \in D} \sum_{j \in S_i^D} \left( \frac{d_i}{P} + \frac{e_{ij}}{R} \right) V_{ij}, \tag{40}$$

$$\sum_{j \in S_i^L} X_{jl} = 1, \quad \forall l \in L, \tag{41}$$

$$\sum_{i \in D_j^S} W_{ij} \leq 1, \quad \forall j \in S, \tag{42}$$

$$\sum_{i \in D_j^S} V_{ij} = \sum_{l \in L_j^S} v_l X_{jl}, \quad \forall j \in S, \tag{43}$$

$$V_{ij} \leq \left( \sum_{l \in L_j^S} v_l \right) W_{ij}, \quad \forall i \in D, \forall j \in S_i^D, \tag{44}$$

$$\sum_{j \in S_i^D} V_{ij} \leq \left( \sum_{l \in L_i^D} v_l \right) Y_i, \quad \forall i \in D, \tag{45}$$

$$X_{jl} \leq \sum_{i \in D_j^S} W_{ij}, \quad \forall j \in S, \quad \forall l \in L_j^S, \tag{46}$$

$$W_{ij} \leq Y_i, \quad \forall i \in D, \quad \forall j \in S_i^D, \tag{47}$$

$$X_{jl} \in \{0, 1\}, \quad \forall j \in S, \quad \forall l \in L_j^S, \tag{48}$$

$$W_{ij} \in \{0, 1\}, \quad \forall i \in D, \quad \forall j \in S_i^D, \tag{49}$$

$$Y_i \in \{0, 1\}, \quad \forall i \in D, \tag{50}$$

$$V_{ij} \geq 0, \quad \forall i \in D, \quad \forall j \in S_i^D. \tag{51}$$

This formulation is a network design problem that involves both continuous flow variables and binary variables on each arc. Solving this model is a difficult task, as demonstrated by the computational experiments reported in Section 8.

A result similar to Proposition 6 for  $BIN(M_{arc})$  also holds for  $BIN(M_{path})$  (we omit the proof, which is essentially the same as that of Proposition 6):

**Proposition 7.** *There exists an optimal solution to  $BIN(M_{path})$  such that*

$$U_j = \frac{1}{Q} \sum_{i \in D_j^S} \sum_{l \in L_j^S} n_l X_{ijl}, \quad \forall j \in S, \tag{52}$$

$$T_i = \frac{1}{P} \sum_{j \in S_i^D} \sum_{l \in L_j^S} v_l X_{ijl}, \quad \forall i \in D, \tag{53}$$

$$H_{ij} = \frac{1}{R} \sum_{l \in L_j^S} v_l X_{ijl}, \quad \forall i \in D, \quad \forall j \in S_i^D. \tag{54}$$

By using this proposition, we can reformulate  $BIN(M_{path})$  as follows:

$$\min \sum_{i \in D} f_i Y_i + \sum_{i \in D} \sum_{j \in S_i^D} \sum_{l \in L_j^S} C_{ijl} X_{ijl}, \tag{55}$$

$$\sum_{j \in S_i^D} \sum_{l \in L_j^S} X_{ijl} = 1, \quad \forall i \in D, \tag{56}$$

$$\sum_{i \in D_j^S} W_{ij} \leq 1, \quad \forall j \in S, \tag{57}$$

$$\sum_{j \in S_i^D} \sum_{l \in L_j^S} v_l X_{ijl} \leq \left( \sum_{l \in L_i^D} v_l \right) Y_i, \quad \forall i \in D, \tag{58}$$

$$X_{ijl} \leq W_{ij}, \quad \forall i \in D, \quad \forall j \in S_i^D, \quad \forall l \in L_j^S, \tag{59}$$

$$W_{ij} \leq Y_i, \quad \forall i \in D, \quad \forall j \in S_i^D, \tag{60}$$

$$X_{ijl} \in \{0, 1\}, \quad \forall i \in D, \quad \forall j \in S_i^D, \quad \forall l \in L_j^S, \tag{61}$$

$$W_{ij} \in \{0, 1\}, \quad \forall i \in D, \quad \forall j \in S_i^D, \tag{62}$$

$$Y_i \in \{0, 1\}, \quad \forall i \in D. \tag{63}$$

In the objective function (55), we have

$$C_{ijl} = c_{jl} + n_l \frac{g_j}{Q} + v_l \left( \frac{d_i}{P} + \frac{e_{ij}}{R} \right), \quad \forall i \in D, \quad \forall j \in S_i^D, \quad \forall l \in L_j^S. \tag{64}$$

This formulation is similar to the path-based version of the uncapacitated two-echelon facility location problem (see the references cited in the Introduction). A major difference between these models and  $BIN(M_{\text{path}})$  is that in the latter there are no fixed costs on the intermediate facilities (here, the satellites). This explains why we can simplify  $BIN(M_{\text{path}})$  and reformulate it as a SPLP, for which large instances can be solved by several efficient algorithms. We formally show this result in the next section.

### 6. SPLP Reformulation of $BIN(M_{\text{path}})$

The transformation of  $BIN(M_{\text{path}})$  into an equivalent SPLP comes from the observation that, in the reformulation derived at the end of the previous section, there are no costs associated to any depot-satellite arc, but only a fixed cost for each depot and a linear cost for each path. For each depot–customer pair  $(i, l)$ , we associate its closest satellite (with ties being broken arbitrarily):

$$j(i, l) = \arg \min_{j \in S_i^D} \left\{ c_{jl} + n_l \frac{g_j}{Q} + v_l \frac{e_{ij}}{R} \right\}, \quad \forall i \in D, \quad l \in L_i^D. \tag{65}$$

We then have the following proposition:

**Proposition 8.** *There exists an optimal solution to  $BIN(M_{\text{path}})$  such that: For any pair  $(i, l)$ ,  $i \in D, l \in L_i^D$ , if  $X_{ij'l} = 1$  for some  $j' \in S_i^D$ , then  $j' = j(i, l)$ .*

**Proof.** The proposition statement follows if we can show that, in any optimal solution to  $BIN(M_{\text{path}})$ , we have: for any pair  $(i, l)$ ,  $i \in D, l \in L_i^D$ , if  $X_{ij'l} = 1$  for some  $j' \in S_i^D$ , then  $c_{j'l} + n_l(g_{j'}/Q) + v_l(e_{ij'}/R) = \min_{j \in S_i^D} \{c_{jl} + n_l(g_j/Q) + v_l(e_{ij}/R)\}$ . Assume the contrary: in some optimal solution  $(W, X, Y)$ , there exist  $i' \in D, l' \in L_{i'}^D$  and  $j' \in S_{i'}^D$  such that  $X_{i'j'l'} = 1$ , but  $c_{j'l'} + n_{l'}(g_{j'}/Q) + v_{l'}(e_{i'j'}/R) > \min_{j \in S_{i'}^D} \{c_{j'l'} + n_{l'}(g_j/Q) + v_{l'}(e_{i'j}/R)\}$ . For notational simplicity, let  $j'' = j(i', l')$ . The hypothesis implies the following inequality:

$$c_{j'l'} + n_{l'} \frac{g_{j'}}{Q} + v_{l'} \frac{e_{i'j'}}{R} > c_{j''l'} + n_{l'} \frac{g_{j''}}{Q} + v_{l'} \frac{e_{i'j''}}{R}.$$

We distinguish two cases:

(1)  $W_{ij''} = 0, \forall i \in D_{j''}^S$

We then define a solution  $(\bar{W}, \bar{X}, Y)$  to  $BIN(M_{\text{path}})$  as follows:

$$\bar{X}_{ijl} = \begin{cases} 0 & \text{if } i = i', j = j', l = l', \\ 1 & \text{if } i = i', j = j'', l = l', \\ X_{ijl} & \text{otherwise;} \end{cases}$$

$$\bar{W}_{ij} = \begin{cases} 0 & \text{if } i = i', j = j', \\ 1 & \text{if } i = i', j = j'', \\ W_{ij} & \text{otherwise.} \end{cases}$$

This solution is feasible, since all customers' requests are satisfied using the same paths as in solution  $(W, X, Y)$ , except for customer  $l'$ , which is now served through path  $(i', j'', l')$  instead of path  $(i', j', l')$ . This solution has an objective

value,  $z(\overline{W}, \overline{X}, Y)$ , strictly lower than the optimal objective value  $z(W, X, Y)$ , since:

$$\begin{aligned} z(W, X, Y) - z(\overline{W}, \overline{X}, Y) &= c_{j'l'} + n_{l'} \frac{g_{j'}}{Q} + v_{l'} \left( \frac{d_{i'}}{P} + \frac{e_{i'j'}}{R} \right) - \left\{ c_{j''l'} + n_{l'} \frac{g_{j''}}{Q} + v_{l'} \left( \frac{d_{i'}}{P} + \frac{e_{i'j''}}{R} \right) \right\} \\ &= c_{j'l'} + n_{l'} \frac{g_{j'}}{Q} + v_{l'} \frac{e_{i'j'}}{R} - \left\{ c_{j''l'} + n_{l'} \frac{g_{j''}}{Q} + v_{l'} \frac{e_{i'j''}}{R} \right\} > 0. \end{aligned}$$

This is a contradiction.

(2) There exists  $i'' \in D_{j''}^S$  such that  $W_{i''j''} = 1$

This case further separates into two subcases:

(2a)  $\frac{d_{i'}}{P} + \frac{e_{i'j''}}{R} > \frac{d_{i''}}{P} + \frac{e_{i''j''}}{R}$

We then define a solution  $(\overline{W}, \overline{X}, Y)$  to  $BIN(M_{\text{path}})$  as follows:

$$\overline{X}_{ijl} = \begin{cases} 0 & \text{if } i = i', j = j', l = l', \\ 1 & \text{if } i = i'', j = j'', l = l', \\ X_{ijl} & \text{otherwise;} \end{cases}$$

$$\overline{W}_{ij} = \begin{cases} 0 & \text{if } i = i', j = j', \\ W_{ij} & \text{otherwise.} \end{cases}$$

This solution is feasible, since all customers' requests are satisfied using the same paths as in solution  $(W, X, Y)$ , except for customer  $l'$ , which is now served through path  $(i'', j'', l')$  instead of path  $(i', j', l')$ . This solution has an objective value,  $z(\overline{W}, \overline{X}, Y)$ , strictly lower than the optimal objective value  $z(W, X, Y)$ , since:

$$\begin{aligned} z(W, X, Y) - z(\overline{W}, \overline{X}, Y) &= c_{j'l'} + n_{l'} \frac{g_{j'}}{Q} + v_{l'} \left( \frac{d_{i'}}{P} + \frac{e_{i'j'}}{R} \right) - \left\{ c_{j''l'} + n_{l'} \frac{g_{j''}}{Q} + v_{l'} \left( \frac{d_{i''}}{P} + \frac{e_{i''j''}}{R} \right) \right\} \\ &> c_{j'l'} + n_{l'} \frac{g_{j'}}{Q} + v_{l'} \left( \frac{d_{i'}}{P} + \frac{e_{i'j'}}{R} \right) - \left\{ c_{j''l'} + n_{l'} \frac{g_{j''}}{Q} + v_{l'} \left( \frac{d_{i'}}{P} + \frac{e_{i'j''}}{R} \right) \right\} \\ &= c_{j'l'} + n_{l'} \frac{g_{j'}}{Q} + v_{l'} \frac{e_{i'j'}}{R} - \left\{ c_{j''l'} + n_{l'} \frac{g_{j''}}{Q} + v_{l'} \frac{e_{i'j''}}{R} \right\} > 0. \end{aligned}$$

Again, this is a contradiction.

(2b)  $d_{i'}/P + e_{i'j''}/R \leq d_{i''}/P + e_{i''j''}/R$

In this case, we define the set  $L(i'', j'') = \{l \in L_{j''}^S \mid X_{i''j''l} = 1\}$ , i.e., the set of customers  $l$  served through path  $(i'', j'', l)$ . We then define a solution  $(\overline{W}, \overline{X}, Y)$  to  $BIN(M_{\text{path}})$  as follows:

$$\overline{X}_{ijl} = \begin{cases} 0 & \text{if } (i = i', j = j', l = l') \text{ or } (i = i'', j = j'', l \in L(i'', j'')), \\ 1 & \text{if } i = i', j = j'', l = l' \text{ or } l \in L(i'', j''), \\ X_{ijl} & \text{otherwise;} \end{cases}$$

$$\overline{W}_{ij} = \begin{cases} 0 & \text{if } (i = i', j = j') \text{ or } (i = i'', j = j''), \\ 1 & \text{if } i = i', j = j'', \\ W_{ij} & \text{otherwise.} \end{cases}$$

This solution is feasible, since all customers' requests are satisfied using the same paths as in solution  $(W, X, Y)$ , except:

for customer  $l'$ , which is now served through path  $(i', j'', l')$  instead of path  $(i', j', l')$ ;

for customers  $l \in L(i'', j'')$ , which are now served through  $(i', j'', l)$  instead of  $(i'', j'', l)$ .

This solution has an objective value,  $z(\bar{W}, \bar{X}, Y)$ , strictly lower than the optimal objective value  $z(W, X, Y)$ , since:

$$\begin{aligned} z(W, X, Y) - z(\bar{W}, \bar{X}, Y) &= c_{j'l'} + n_{l'} \frac{g_{j'}}{Q} + v_{l'} \left( \frac{d_{i'}}{P} + \frac{e_{i'j'}}{R} \right) \\ &\quad + \sum_{l \in L(i'', j'')} \left\{ c_{j''l} + n_l \frac{g_{j''}}{Q} + v_l \left( \frac{d_{i''}}{P} + \frac{e_{i''j''}}{R} \right) \right\} \\ &\quad - \left\{ c_{j''l'} + n_{l'} \frac{g_{j''}}{Q} + v_{l'} \left( \frac{d_{i'}}{P} + \frac{e_{i'j''}}{R} \right) \right\} \\ &\quad - \sum_{l \in L(i'', j'')} \left\{ c_{j''l} + n_l \frac{g_{j''}}{Q} + v_l \left( \frac{d_{i'}}{P} + \frac{e_{i'j''}}{R} \right) \right\} \\ &= c_{j'l'} + n_{l'} \frac{g_{j'}}{Q} + v_{l'} \frac{e_{i'j'}}{R} - \left\{ c_{j''l'} + n_{l'} \frac{g_{j''}}{Q} + v_{l'} \frac{e_{i'j''}}{R} \right\} \\ &\quad + \sum_{l \in L(i'', j'')} v_l \left\{ \left( \frac{d_{i''}}{P} + \frac{e_{i''j''}}{R} \right) - \left( \frac{d_{i'}}{P} + \frac{e_{i'j''}}{R} \right) \right\} > 0. \end{aligned}$$

In this case also, we have a contradiction.  $\square$

We use the last proposition to define the following variables:

$$X_{il} = X_{ij(i,l)l}, \quad \forall i \in D, \quad \forall l \in L_i^D. \tag{66}$$

The SPLP reformulation of the path-based binary relaxation, denoted  $M_{\text{splp}}$ , is then written as follows:

$$\min \sum_{i \in D} f_i Y_i + \sum_{i \in D} \sum_{l \in L_i^D} C_{il} X_{il} \tag{67}$$

$$\sum_{i \in D} \sum_{l \in L_i^D} X_{il} = 1, \quad \forall l \in L, \tag{68}$$

$$X_{il} \leq Y_i, \quad \forall i \in D, \quad \forall l \in L_i^D, \tag{69}$$

$$X_{il} \in \{0, 1\}, \quad \forall i \in D, \quad \forall l \in L_i^D, \tag{70}$$

$$Y_i \in \{0, 1\}, \quad \forall i \in D. \tag{71}$$

In the objective function (67), we have

$$C_{il} = C_{ij(i,l)l}, \quad \forall i \in D, \quad \forall l \in L_i^D. \tag{72}$$

**Proposition 9.**  $M_{\text{splp}}$  is equivalent to  $BIN(M_{\text{path}})$ .

**Proof.** (1)  $Z(BIN(M_{\text{path}})) \geq Z(M_{\text{splp}})$ .

Consider an optimal solution to  $BIN(M_{\text{path}})$  and assume, without loss of generality, that this solution satisfies the conditions of Proposition 8. From (66), we have:

$$(a) \quad X_{il} = X_{ij(i,l)l} = \sum_{j \in S_i^D} X_{ijl}, \quad i \in D, \quad l \in L_i^D.$$

We can then derive a solution to  $M_{\text{splp}}$ , the  $Y$  variables retaining the same values as in the optimal solution to  $BIN(M_{\text{path}})$ ; this solution is feasible for  $M_{\text{splp}}$  since:

by using equations (a), we derive constraints (68) from (56);

constraints (69) are satisfied, as otherwise this would imply that for some pair  $(i', l')$ ,  $i' \in D$ ,  $l' \in L_{i'}^D$ , we would have  $X_{i'l'} = X_{i'j(i',l')l'} = 1$  and  $Y_{i'} = 0$ , but this would contradict constraints (59) and (60).

By equations (a), this solution has the same objective value as the optimal solution to  $BIN(M_{\text{path}})$ , which shows that  $Z(BIN(M_{\text{path}})) \geq Z(M_{\text{splp}})$ .

(2)  $Z(M_{\text{splp}}) \geq Z(BIN(M_{\text{path}}))$

Consider an optimal solution to  $M_{\text{splp}}$  and define

$$X_{ijl} = \begin{cases} X_{il} & \text{if } j = j(i, l), i \in D, j \in S_i^D, l \in L_j^S, \\ 0 & \text{otherwise;} \end{cases}$$

$$W_{ij} = \max_{l \in L_j^S, j=j(i,l)} \{X_{il}\}, \quad i \in D, \quad j \in S_i^D.$$

These definitions imply the two following equations:

- (a)  $X_{il} = X_{ij(i,l)} = \sum_{j \in S_i^D} X_{ijl}, i \in D, l \in L_i^D;$
- (b)  $W_{ij} = \max_{l \in L_j^S} \{X_{ijl}\}, i \in D, j \in S_i^D.$

We can then derive a solution to  $BIN(M_{\text{path}})$ , the  $Y$  variables retaining the same values as in the optimal solution to  $(M_{\text{splp}})$ ; with the exception of (57), this solution satisfies all constraints of  $BIN(M_{\text{path}})$ :

by using equations (a), we derive constraints (56) from (68);

equations (b) imply constraints (59);

by combining the definition of the  $W$  variables with constraints (69), we obtain constraints (60);

constraints (58) are implied by (59) and (60).

By equations (a), this solution has the same objective value as that of the optimal solution to  $M_{\text{splp}}$ .

It remains to show that constraints (57) are satisfied by the solution we have defined, or that we can always derive from it a solution with the same objective value that satisfies them. Let us assume that constraints (57) are not satisfied by the solution we have defined. This implies that there exists some  $j' \in S$  such that there at least two pairs  $(i', l')$  and  $(i'', l'')$  with  $i' \neq i'', l' \neq l'', j' = j(i', l') = j(i'', l'')$  and  $X_{i'l'} = X_{i''l''} = 1$ . Without loss of generality, let us assume that  $C_{i'l'} \leq C_{i''l''}$ . We then define a feasible solution  $(\bar{X}, Y)$  as follows:

$$\bar{X}_{il} = \begin{cases} X_{il}, & l \neq l'', i \in D | l \in L_i^D, \\ 1, & l = l'', i = i', \\ 0, & l = l'', i = i''. \end{cases}$$

This solution is feasible to  $M_{\text{splp}}$ , since only the values of the  $X$  variables for the pairs  $(i', l')$  and  $(i'', l'')$  have been modified, and

- (1)  $\sum_{i \in D | l'' \in L_i^D} \bar{X}_{il''} = \bar{X}_{i'l''} = 1;$
- (2)  $\bar{X}_{i'l''} = 1 = Y_{i'};$
- (3)  $\bar{X}_{i''l''} = 0 < Y_{i''}.$

This solution has an objective value,  $z(\bar{X}, Y)$ , no greater than the optimal objective value  $z(X, Y)$ , since:

$z(X, Y) - z(\bar{X}, Y) = C_{i''l''} X_{i''l''} - C_{i'l'} \bar{X}_{i'l''} = C_{i''l''} - C_{i'l'} \geq 0$ . If  $C_{i''l''} > C_{i'l'}$ , we have a contradiction, which implies that constraints (57) are satisfied. If  $C_{i''l''} = C_{i'l'}$ , solution  $(\bar{X}, Y)$  has the same objective value as that of  $(X, Y)$ , but has one less depot connected to satellite  $j'$ . By repeating the same argument a finite number of times, either we obtain a contradiction, or we derive another solution to  $M_{\text{splp}}$  having the same objective value and satisfying constraints (57). □

Although  $M_{\text{splp}}$  is an attractive reformulation of  $BIN(M_{\text{path}})$ , it is still an NP-hard problem. Hence, we might be interested in assessing its LP relaxation and comparing it to  $LP(M_{\text{path}})$ .

**Proposition 10.**  $LP(M_{\text{splp}})$  dominates  $LP(M_{\text{path}})$ .

**Proof.** Consider an optimal solution  $(X, Y)$  to  $LP(M_{\text{splp}})$  and derive a solution to  $LP(M_{\text{path}})$  in the same way as in the second part of the proof of the previous proposition, where it was shown that  $Z(M_{\text{splp}}) \geq Z(\text{BIN}(M_{\text{path}}))$ . Equations (a) and (b) are also satisfied and:

- by using equations (a), we derive constraints (56) from (68);
- equations (b) imply constraints (59);
- by combining the definition of the  $W$  variables with constraints (69), we obtain constraints (60);
- constraints (58) are satisfied, since we have, for each  $i \in D$

$$\begin{aligned} \sum_{j \in S_i^D} \sum_{l \in L_j^S} v_l X_{ijl} &= \sum_{l \in L_j^S, j=j(i,l)} v_l X_{il} \\ &\leq \sum_{l \in L_i^D} v_l X_{il} \\ &\leq \left( \sum_{l \in L_i^D} v_l \right) \max_{l \in L_i^D} \{X_{il}\} \\ &= \left( \sum_{l \in L_i^D} v_l \right) Y_i. \end{aligned}$$

By equations (a), this solution has the same objective value as that of the optimal solution to  $M_{\text{splp}}$ .

It remains to show that constraints (57) are satisfied by the solution we have defined, or that we can always derive from it a solution with the same objective value that satisfies them. Let us assume that constraints (57) are not satisfied by the solution we have defined. This implies that there exists some  $j' \in S$  such that there at least two pairs  $(i', l')$  and  $(i'', l'')$  with  $i' \neq i'', j' = j(i', l') = j(i'', l'')$  and

$$X_{i'l'} + X_{i''l''} > 1. \tag{73}$$

Note that we also have

$$X_{i'l'} + X_{i''l''} - (Y_{i'} - X_{i'l''}) \leq 1, \tag{74}$$

since

$$\begin{aligned} X_{i'l'} + X_{i''l''} - (Y_{i'} - X_{i'l''}) &\leq Y_{i'} + X_{i''l''} - (Y_{i'} - X_{i'l''}) \\ &= X_{i'l''} + X_{i''l''} \\ &\leq 1, \end{aligned}$$

by constraints (68). Inequalities (73) and (74) imply

$$X_{i'l''} < Y_{i'}. \tag{75}$$

Without loss of generality, let us assume that  $C_{i'l''} \leq C_{i''l''}$ . We then define a feasible solution  $(\bar{X}, Y)$  as follows:

$$\bar{X}_{il} = \begin{cases} X_{il}, & l \neq l'', i \in D | l \in L_i^D, \\ Y_{i'}, & l = l'', i = i', \\ X_{i''l''} - (Y_{i'} - X_{i'l''}), & l = l'', i = i''. \end{cases}$$

This solution is feasible for  $LP(M_{\text{splp}})$ , since only the values of the  $X$  variables for the pairs  $(i', l'')$  and  $(i'', l'')$  have been modified, and

- (1)  $\sum_{i \in D_{i''}^L} \bar{X}_{il''} = \sum_{i \in D_{i''}^L, i \neq i', i''} X_{il''} + Y_{i'} + (X_{i''l''} - (Y_{i'} - X_{i'l''})) = \sum_{i \in D_{i''}^L} X_{il''} = 1$ ;
- (2)  $\bar{X}_{i'l''} = Y_{i'}$ ;
- (3)  $\bar{X}_{i''l''} = X_{i''l''} - (Y_{i'} - X_{i'l''}) < X_{i'l''} \leq Y_{i''}$ .



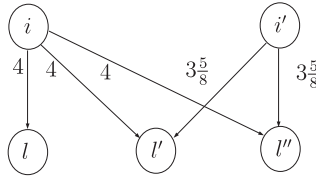


Fig. 5. SPLP reformulation of the problem instance of Fig. 3.

This solution has an objective value,  $z(\bar{X}, Y)$ , no greater than the optimal objective value  $z(X, Y)$  since:

$z(X, Y) - z(\bar{X}, Y) = C_{i'l''} X_{i'l''} + C_{i'l''} X_{i'l''} - C_{i'l''} Y_{i'} - C_{i'l''} \{X_{i'l''} - (Y_{i'} - X_{i'l''})\} = (C_{i'l''} - C_{i'l''})(Y_{i'} - X_{i'l''}) \geq 0$ . If  $C_{i'l''} > C_{i'l''}$ , we obtain  $(C_{i'l''} - C_{i'l''})(Y_{i'} - X_{i'l''}) > 0$  by (75), which leads to a contradiction and implies that constraints (57) are satisfied. If  $C_{i'l''} = C_{i'l''}$ , solution  $(\bar{X}, Y)$  has the same objective value as that of  $(X, Y)$ , but (74) implies that  $\bar{X}_{i'l'} + \bar{X}_{i'l''} = X_{i'l'} + X_{i'l''} - (Y_{i'} - X_{i'l''}) \leq 1$ . By repeating the same argument a finite number of times, either we obtain a contradiction, or we derive another solution to  $LP(M_{splp})$  having the same objective value and satisfying constraints (57).

To show an example where  $Z(LP(M_{splp})) > Z(LP(M_{path}))$ , consider again the problem instance illustrated in Fig. 3. When reformulated as an SPLP, this problem instance appears as in Fig. 5. In an optimal solution to this problem, only depot  $i$  is open, with an objective value equal to  $Z(LP(M_{splp})) = 13 > 12 \frac{3}{4} = Z(LP(M_{path}))$ . □

In the last example,  $LP(M_{splp})$  provides an integral optimal solution. Unfortunately, this is not always the case. Krarup and Pruzan [16] give such an example, due to Gomory, where the LP relaxation of the SPLP formulation provides a fractional solution. However, it has been observed, using many different types of instances, that the LP relaxation of the SPLP often provides an integral optimal solution. This observation is also verified by our computational experiments reported in Section 8.

### 7. Summary of bound relationships

The following proposition summarizes the relationships between the different formulations and relaxations introduced so far.

**Proposition 11.** *The following relationships hold between the optimal values of the different relaxations, where the “ $\geq$ ” sign means a relation of dominance:*

$$\begin{aligned} Z(M_{path}) &= Z(M_{arc}) \\ &\geq Z(BIN(M_{arc})) \\ &= Z(BIN(M_{path})) \\ &= Z(M_{splp}) \\ &\geq Z(LP(M_{splp})) \\ &\geq Z(LP(M_{path})) \\ &\geq Z(LP(M_{arc})). \end{aligned}$$

**Proof.** The equation on the first line has been shown in Proposition 3.

The inequality on the second line follows immediately by definition of the binary relaxation of  $M_{arc}$ . Moreover, the inequality can be strict, since integer solutions to  $M_{arc}$  do not necessarily satisfy the equations in the statement of Proposition 6.

The equation on the third line corresponds to Proposition 5.

The equation on the fourth line has been shown in Proposition 9.

The inequality on the fifth line follows immediately by definition of the LP relaxation of  $M_{splp}$ . Moreover, the inequality can be strict, as we can easily derive an instance of our problem for which the LP relaxation of the SPLP reformulation provides a fractional optimal solution.

The inequality on the sixth line has been shown in Proposition 10. The inequality can be strict, as shown by the problem instance of Fig. 3.

Finally, the last inequality corresponds to Proposition 4. Again, the problem instance of Fig. 3 shows that the inequality can be strict.  $\square$

## 8. Computational results

The main objective of our computational experiments is to compare the relaxations and formulations on data obtained from a major French mail-order company, which provided us with a typical network having 93 depots, 320 satellites and 701 customers. The company also provided us with realistic estimates of the costs and the capacities. Based on this real-application data, we have generated 32 instances by specifying:

- subsets of the sets of depots, satellites and customers, i.e.,  $D$ ,  $S$  and  $L$  (three subnetworks, large, medium and small, were generated);
- multipliers  $M_f$ ,  $M_g$  and  $M_p$  for, respectively, the fixed costs at the depots, the unit batch costs at the satellites and the capacities of the medium-size vehicles at the depots (two values, 1 and 2, were tested for each multiplier).

Every instance is denoted  $X(M_f, M_g, M_p)$ , with  $X = R, L, M$  or  $S$ , standing for real-application, large-scale, medium-scale or small-scale network,  $M_f$ ,  $M_g$  and  $M_p$  denoting the multiplier values. Table 1 summarizes the characteristics of the 32 instances. Column 1 gives the problem name, while the next three columns indicate the number of depots, satellites and customers. The next two columns show, respectively, the number of arcs between depots and satellites, denoted  $|A_{DS}|$ , and the number of arcs between satellites and customers, denoted  $|A_{SL}|$ .

Our experiments consist in solving all relaxations and formulations with the state-of-the-art LP/MIP solver CPLEX (version 10.0) on an AMD Opteron 248 operating at 2.2 Ghz and equipped with 4 Gig RAM (the operating system is Centos 4.2, Linux-RedHat). We use the default CPLEX parameters to solve the LP relaxations and the MIP formulations, except that we impose a limit of 2 h of CPU time.

To analyze our computational results, we use the CPU time in seconds, which we denote CPU, and the lower bound gap (%)  $\Delta Z^L = 100 \times (Z^* - Z^L)/Z^*$  between the lower bound  $Z^L$  corresponding to a particular relaxation and the best known upper bound  $Z^*$ , obtained by solving the arc-based and path-based formulations with CPLEX for a maximum of 2 h each. For the arc-based and path-based formulations,  $\Delta Z^L$  measures the gap between  $Z^*$  and the best lower bound obtained by CPLEX after 2 h of CPU time. For the arc-based formulation, we also display the upper bound gap (%)  $\Delta Z^U = 100 \times (Z^U - Z^*)/Z^*$  between the best known upper bound  $Z^*$  and the upper bound  $Z^U$  obtained by CPLEX when solving the arc-based model for a limit of 2 h. For the path-based model, we could show the same measure, but  $\Delta Z^U = 0$  for  $M_{\text{path}}$  on all instances, except one (we further comment on this result below). Table 2 presents these measures for the real-application and large-scale networks, while Table 3 displays the results obtained with the medium-scale and small-scale networks.

Several conclusions can be drawn from these results:

- The bound relationships summarized in Proposition 11 are verified for all instances, with the exception that for most of the real-application and large-scale instances, we observe a gap between the lower bounds computed for relaxations  $BIN(M_{\text{arc}})$  and  $BIN(M_{\text{path}})$ : this is explained by the fact that CPLEX could not solve  $BIN(M_{\text{arc}})$  to optimality within 2 h (7200 s) of CPU time.

Table 1  
Set of 32 instances (each row contains eight instances).

Problem	$ D $	$ S $	$ L $	$ A_{DS} $	$ A_{SL} $	$M_f$	$M_g$	$M_p$
$R(M_f, M_g, M_p)$	93	320	701	2250	28782	{1, 2}	{1, 2}	{1, 2}
$L(M_f, M_g, M_p)$	70	240	526	1260	16131	{1, 2}	{1, 2}	{1, 2}
$M(M_f, M_g, M_p)$	46	160	350	562	6652	{1, 2}	{1, 2}	{1, 2}
$S(M_f, M_g, M_p)$	23	80	175	167	1807	{1, 2}	{1, 2}	{1, 2}

Table 2  
Performance analysis for real-application and large-scale instances.

Problem	$LP(M_{arc})$	$LP(M_{path})$	$LP(M_{splp})$	$BIN(M_{arc})$	$BIN(M_{path})$	$M_{splp}$	$M_{arc}$		$M_{path}$
	$\Delta Z^L$ CPU	$\Delta Z^L$ CPU	$\Delta Z^L$ CPU	$\Delta Z^L$ CPU	$\Delta Z^L$ CPU	$\Delta Z^L$ CPU	$\Delta Z^L$ CPU	$\Delta Z^U$	$\Delta Z^L$ CPU
R(1,1,1)	17.2 6	15.6 10	14.5 1	15.5 7200	14.5 110	14.5 1	8.5 7200	5.3	1.1 7200
R(1,1,2)	18.7 6	18.0 14	16.9 1	17.1 7200	16.9 99	16.9 1	10.1 7200	12.2	1.1 7200
R(1,2,1)	17.6 6	16.2 18	15.2 1	16.1 7200	15.2 117	15.2 1	9.9 7200	4.0	3.2 7200
R(1,2,2)	17.8 7	17.1 14	16.2 1	16.4 7200	16.2 99	16.2 1	10.1 7200	8.8	1.7 7200
R(2,1,1)	17.4 7	16.1 13	12.9 1	15.0 7200	12.9 95	12.9 1	9.3 7200	6.2	1.4 7200
R(2,1,2)	18.7 8	18.1 12	14.8 1	16.0 7200	14.8 91	14.8 1	10.3 7200	12.1	1.1 7200
R(2,2,1)	16.2 7	15.0 13	12.0 1	13.3 7200	12.0 97	12.0 1	8.4 7200	4.3	1.3 7200
R(2,2,2)	18.5 7	18.0 12	15.0 1	16.0 7200	15.0 94	15.0 1	10.9 7200	9.1	2.6 7200
L(1,1,1)	15.2 8	14.0 3	12.1 0	12.3 7200	12.1 27	12.1 1	7.4 7200	3.1	0.6 7200
L(1,1,2)	16.5 10	15.9 6	13.1 0	13.2 7200	13.0 117	13.0 1	9.4 7200	4.2	1.0 7200
L(1,2,1)	14.5 7	13.4 6	11.6 0	11.7 7200	11.6 27	11.6 1	7.0 7200	4.7	1.0 7200
L(1,2,2)	15.3 10	14.8 6	12.1 0	12.1 7200	12.0 139	12.0 6	8.3 7200	3.4	0.9 7200
L(2,1,1)	17.0 8	15.8 7	10.7 0	10.7 5120	10.7 33	10.7 1	7.5 7200	2.1	0.5 7200
L(2,1,2)	19.5 7	18.9 8	12.7 0	13.1 7200	12.7 55	12.7 1	9.8 7200	1.5	1.9 7200
L(2,2,1)	16.0 8	14.8 7	10.0 0	10.0 1997	10.0 36	10.0 1	8.4 7200	4.3	0.6 7200
L(2,2,2)	18.2 7	17.6 8	11.8 0	11.8 2830	11.8 64	11.8 1	8.5 7200	1.8	1.5 7200

- The bound  $Z(LP(M_{path}))$  improves slightly upon  $Z(LP(M_{arc}))$  (0.7% on average), while  $Z(LP(M_{splp}))$  improves significantly upon  $Z(LP(M_{path}))$ , with an average improvement of 2.8% (3.8% when the depot fixed costs are doubled).
- On almost all instances, except five, we have  $Z(LP(M_{splp})) = Z(M_{splp})$ . Even when there is a gap between the two bounds, it is always less than 1%. These results are consistent with similar ones reported in the literature on the SPLP.
- When the depot fixed costs are doubled, the gaps for the SPLP relaxation and for its LP relaxation always decrease, while no clear trend can be observed for the other LP relaxations.
- When the capacities of the medium-size trucks are doubled, the gaps always increase for all relaxations on all instances (by more than 3% on average). No significant tendency can be observed when the satellite unit batch costs are doubled.
- On all instances,  $BIN(M_{path})$  can be solved in reasonable CPU times (always less than 3 min), but the same bound can always be computed in 1 s or less by solving  $M_{splp}$ .
- The arc-based and path-based formulations can be solved to optimality only for the small-scale instances, for which CPLEX solves  $M_{arc}$  more efficiently than  $M_{path}$ . For all other instances, CPLEX provides better results when solving  $M_{path}$  within the limit of 2 h of CPU time: for all instances, but one, M(2, 2, 1), the best known solution is

Table 3  
Performance analysis for medium-scale and small-scale instances.

Problem	$LP(M_{\text{arc}})$	$LP(M_{\text{path}})$	$LP(M_{\text{splp}})$	$BIN(M_{\text{arc}})$	$BIN(M_{\text{path}})$	$M_{\text{splp}}$	$M_{\text{arc}}$		$M_{\text{path}}$
	$\Delta Z^L$ CPU	$\Delta Z^L$ CPU	$\Delta Z^L$ CPU	$\Delta Z^L$ CPU	$\Delta Z^L$ CPU	$\Delta Z^L$ CPU	$\Delta Z^L$ CPU	$\Delta Z^U$	$\Delta Z^L$ CPU
M(1,1,1)	13.3 1	12.8 0	11.3 0	11.3 22	11.3 2	11.3 0	4.0 7200	0.3	0.8 7200
M(1,1,2)	15.1 1	14.8 1	13.1 0	13.1 17	13.1 3	13.1 1	3.5 7200	0.5	0.7 7200
M(1,2,1)	13.0 1	12.4 1	11.1 0	11.1 25	11.1 2	11.1 1	4.3 7200	1.0	1.1 7200
M(1,2,2)	14.6 1	14.3 1	12.7 0	12.7 14	12.7 2	12.7 1	3.8 7200	1.9	1.0 7200
M(2,1,1)	14.2 1	13.8 1	10.0 0	9.6 29	9.6 7	9.6 1	3.8 7200	0.0	0.8 7200
M(2,1,2)	16.0 1	15.8 1	11.7 0	11.1 21	11.1 17	11.1 1	4.0 7200	0.1	0.6 7200
M(2,2,1)	14.2 1	13.8 1	10.3 0	9.9 25	9.9 7	9.9 1	4.4 7200	0.0	1.6 7200
M(2,2,2)	15.6 1	15.4 1	11.6 0	11.0 25	11.0 9	11.0 1	4.2 7200	0.3	1.1 7200
S(1,1,1)	16.2 0	15.8 0	14.0 0	14.0 1	14.0 1	14.0 0	0.0 1998	0.0	0.0 3636
S(1,1,2)	18.5 0	18.4 0	16.3 0	16.3 1	16.3 1	16.3 1	0.0 1026	0.0	0.0 3487
S(1,2,1)	16.9 0	16.5 0	14.8 0	14.8 1	14.8 1	14.8 1	0.0 940	0.0	0.0 5348
S(1,2,2)	19.0 0	18.9 0	16.9 0	16.9 1	16.9 1	16.9 1	0.0 1812	0.0	0.0 2598
S(2,1,1)	15.2 0	15.0 0	11.3 0	11.3 1	11.3 1	11.3 1	0.0 612	0.0	0.0 1970
S(2,1,2)	17.2 0	17.1 0	13.2 0	13.2 1	13.2 1	13.2 1	0.0 266	0.0	0.0 5624
S(2,2,1)	15.8 0	15.6 0	12.1 0	12.1 1	12.1 1	12.1 1	0.0 2678	0.0	0.0 4198
S(2,2,2)	17.6 0	17.5 0	13.9 0	13.9 1	13.9 1	13.9 1	0.0 1635	0.0	0.0 4225

obtained when solving  $M_{\text{path}}$  (for instance M(2, 2, 1),  $\Delta Z^U = 0.5$  for  $M_{\text{path}}$ ). For the real-application and large-scale instances, CPLEX always identifies the best solution when solving  $M_{\text{path}}$  with an average upper bound gap  $\Delta Z^U = 5.4$  for  $M_{\text{arc}}$ .

## 9. Conclusion

In this paper, we have discussed several formulations and relaxations for a multi-echelon location–distribution problem arising from an actual application in fast delivery service. We have shown that the LP relaxation of a path-based formulation ( $M_{\text{path}}$ ) provides a better bound than the LP relaxation of an arc-based model ( $M_{\text{arc}}$ ). We have shown that the binary relaxations of the two formulations ( $BIN(M_{\text{path}})$  and  $BIN(M_{\text{arc}})$ ) provide the same bound, but that the path-based binary relaxation appears preferable from a computational point of view, since it can be reformulated as an equivalent simple plant location problem ( $M_{\text{splp}}$ ). We have also shown that the LP relaxation of  $M_{\text{splp}}$  provides a better bound than the LP relaxation of  $M_{\text{path}}$ . We have presented computational results on solving with CPLEX the different formulations and relaxations on a large-scale application and on several instances derived from this real application. These results have demonstrated that the LP relaxation of  $M_{\text{splp}}$  is effective in reducing the gap compared to the other LP relaxations. Moreover, it can be solved very efficiently by CPLEX, even for large-scale instances. These results

have also shown that CPLEX (with a reasonable limit of CPU time) generally provides better solutions when solving formulation  $M_{\text{path}}$  rather than  $M_{\text{arc}}$ .

Although CPLEX, with a reasonable CPU limit, could find feasible solutions within 2% for almost all instances, it could not prove optimality easily for the largest instances. Therefore, a promising research avenue is to explore the development of exact branch-and-bound algorithms based on the SPLP reformulation of the path-based model. These algorithms would gain efficiency if they are combined with heuristics inspired by local search principles. Other promising research avenues consist in studying extensions to our problem. In particular, we have mentioned in the Introduction that our formulations do not integrate any routing aspect, the latter being taken into account in a preprocessing phase. In real applications, both the location–distribution and the routing decisions must be handled simultaneously. Major potential savings can be obtained by considering these two aspects together. Finally, the problem we consider is for a static short-term horizon (typically, 24 h). It would be interesting to consider dynamic versions of the problem, thus allowing to extend the problem to a medium-term horizon (typically, one week).

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